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Chekanov–Eliashberg invariant of Legendrian knots: existence of augmentations

Dmitry Fuchs

Department of Mathematics, University of California, One Shields Ave., Davis, CA 95616, USA

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Abstract

An explicit and easy-to-check condition sufficient for the existence of an augmentation in the Chekanov–Eliashberg algebra of a Legendrian knot is given in terms of the front diagram. Many new examples of Legendrian knots distinguishable by the linearized version of the Chekanov–Eliashberg invariant are provided.

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Subj. Class: Differential geometry

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1. Introduction

In this paper, we consider Legendrian knots in the standard 3-dimensional contact space. The main goal of this work is to make the new invariant of these knots, recently constructed by Chekanov [3] and Eliashberg [4], work in a more effective way.

A working version of the Chekanov–Eliashberg invariant is based on a linearization of their complex by means of an *augmentation* as defined by Chekanov [3]. If an augmentation exists, then there arises a chance, almost without computations, to distinguish Legendrian knots, indistinguishable by more traditional invariants; it should be added, that, so far, it is not known whether the non-linearized version of this invariant actually distinguishes Legendrian knots indistinguishable by the linearized version.

Technically, the main result of this work is an easy-to-check sufficient condition on the existence of an augmentation. As an application, we will construct new examples of Legendrian non-isotopic Legendrian knots.

E-mail address: fuchs@math.ucdavis.edu (D. Fuchs).

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1.1. Knots

A Legendrian knot in the standard 3-dimensional contact space is a smooth, non-self-intersecting, closed curve in $\mathbb{R}^3$ with zero restriction of the standard contact form $y\,dx - dz$. A Legendrian isotopy is an isotopy via Legendrian knots. Any topological isotopy class of knots contains infinitely many Legendrian isotopy classes of Legendrian knots, so the Legendrian isotopy classification of Legendrian knots is finer than the topological isotopy classification of knots.

1.2. Diagrams

There exist two diagrammatic presentations of Legendrian knots: by *xz*-diagrams, or front diagrams, and by *xy*-diagrams. A generic *xz*-projection of a Legendrian knot is a closed smooth curve with cusps and transverse double crossings, without other singularities (in particular, without self-tangencies), and without vertical tangents. The *xz*-projection determines a Legendrian knot uniquely: the missing y-coordinate is the slope of the curve. A generic *xy*-projection of a Legendrian knot is a smooth curve with transverse double crossings, without other singularities, satisfying the following area conditions: (1) the total area bounded by the curve is 0, (2) each of the two pieces, into which the curve is cut by any crossing, bounds a non-zero area. The *xy*-projection determines a Legendrian knot up to a translation parallel to the z-axis: the difference between the z-coordinates of two points of the knot is equal to the integral of $y\,dx$ over the segment of the *xy*-diagram between the projections of these two points.

It should be noted that the non-local area conditions make *xy*-diagrams much less convenient than front diagrams. In particular, no Reidemeister-like description of admissible Legendrian moves exists for *xy*-diagrams. (For front diagrams such a description is well known [15]; it can also be found in many works in the bibliography [16,12,8], etc.)

1.3. Classical invariants

There are two classical integral invariants of Legendrian knots which may distinguish Legendrian isotopy classes within a topological isotopy class: the Maslov (or rotation) number and the Thurston–Bennequin number [1]; the first is defined for oriented Legendrian knots and changes sign when the orientation is reversed, the second does not depend on the orientation. These numbers are determined by the (oriented in the case of the Maslov invariant) Legendrian isotopy type and can be easily determined from either the *xy*- or *xz*-diagram. In terms of an *xy*-diagram, the Maslov number is the rotation number of the curve, while the Thurston–Bennequin number is the writhe (see [13]). For an oriented *xz*-diagram, we denote by p the number of crossings with the same horizontal directions of the strands (recall, that vertical tangents are not allowed), by m the number of crossings with the opposite horizontal directions of the strands, by c the total number of cusps (the numbers p, m, c do not depend on the orientation), by u the number of upward cusps (i.e., the lower strand is directed to the cusp, and the upper strand is directed from the cusp), and by d the number of downward cusps (thus, $c = u + d$). Then the Maslov number is

$\frac{1}{2}(d-u)$, and the Thurston–Bennequin number is $p-m-\frac{1}{2}c$ (see [1,16,12], etc.). we use the notations $\tau\beta$ and μ for the Thurston–Bennequin and Maslov numbers below.

1.4. *Values of Thurston–Bennequin numbers*

It is easy to lower the Thurston–Bennequin number of a Legendrian knot within a topological isotopy class: just insert a Z-shaped zigzag (composed of two cusps) anyplace. It is more difficult to raise this number. Actually, at least three upper estimates of the Thurston–Bennequin number within a topological isotopy class are known. Let Γ be a Legendrian knot, S be a Seifert surface bounded by Γ, $e(\Gamma)$ be the highest degree of α in the HOMFLY polynomial $P_\Gamma(\alpha, z)$ of the knot Γ, and $f(\Gamma)$ be the highest degree of a in the Kauffman polynomial $F_\Gamma(a, z)$ of Γ (we use the notation of [13]).

Theorem 1.1 (Bennequin [1]). $\tau\beta(\Gamma)+|\mu(\Gamma)|\leq-\chi(S)$ (*where* χ *is the Euler characteristic*).

Theorem 1.2 (Fuchs and Tabachnikov [12]). $\tau\beta(\Gamma)+|\mu(\Gamma)|<-e(\Gamma)$.

Theorem 1.3 (Fuchs and Tabachnikov [12], Chmutov and Goryunov [2] and Tabachnikov [17]). $\tau\beta(\Gamma)<-f(\Gamma)$.

None of these results gives an exact estimate for either $\tau\beta(\Gamma)$ or $\tau\beta(\Gamma)+|\mu(\Gamma)|$ (see [7,9,11,12]), and no one of them implies any of the others (see [11]).

1.5. *Sufficiency of classical invariants*

For some topological isotopy classes of knots, it is known that any two oriented Legendrian knots of this class having equal Thurston–Bennequin and Maslov numbers are oriented Legendrian isotopic. It is proved for topological unknots by Eliashberg and Fraser [5] and for torus knots by Etnyre and Honda [10].

1.6. *Chekanov–Eliashberg invariants*

It is not true, however, that topologically isotopic oriented Legendrian knots with equal Thurston–Bennequin and Maslov numbers are always Legendrian isotopic. A new invariant that may distinguish such knots was constructed in 1997 independently by Chekanov [3] and Eliashberg [4]. (Eliashberg's construction was based on a contact version of Floer cohomology, worked out jointly with Hofer. A detailed presentation of this version can be found, among other things, in a recent paper by Eliashberg et al. [6].) Both Chekanov and Eliashberg were able to exhibit an (actually, the same) example of a pair of topologically isotopic, Legendrian non-isotopic oriented Legendrian knots, which were not distinguishable by the previously known invariants. We will not discuss the true origin of this invariant, but rather recall its construction, as given in the original works of the authors.

Let Γ be an oriented Legendrian knot presented by a generic xy-diagram. At any crossing $\mathbf{x}$, the two strands form four angles, of which we call two positive and two negative: the positive angles are bounded on the right by the upper strand and on the left by the lower strand. For $n \geq 0$, we fix a convex planar domain P_n, bounded by a piecewise smooth curve Π_n of $n+1$ vertices, which are denoted in a counterclockwise direction as $u_0, u_1, \dots, u_n$; when $n \geq 2$, P_n may be a regular $(n+1)$-gon. We consider orientation preserving immersions f of P_n into the xy-plane, which map Π_n into the diagram of Γ in such a way that each u_i is mapped onto a crossing. Additionally, we suppose that a neighborhood of u_0 is mapped onto a positive angle, while the neighborhoods of $u_1, \dots, u_n$ are mapped onto negative angles. For any n, the set of isotopy classes of such immersions f with $f(u_0) = \mathbf{x}$ is denoted as $\mathrm{Imm}_n(\mathbf{x})$. For any $\mathbf{x}$, $\cup_{n=0}^{\infty} \mathrm{Imm}_n(\mathbf{x})$ is finite (because, for any $f \in \mathrm{Imm}_n(\mathbf{x})$, $h(\mathbf{x}) - \sum_{i=1}^{n} h(f(u_i)) > 0$, where $h(\mathbf{x})$ is the absolute value of the difference of the z-coordinates of the points above the crossing $\mathbf{x}$). Let $\mathbf{A}$ denote the differential $\mathbb{Z}_2$-algebra, which is defined as the free associative algebra generated by the crossings of the xy-diagram of Γ, whose differential $d : \mathbf{A} \to \mathbf{A}$ satisfies the product rule and the formula $d(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{f \in \mathrm{Imm}_n(\boldsymbol{x})} f(u_1) \cdots f(u_n)$.

Theorem 1.4 (Chekanov [3] and Eliashberg [4]). *If* $\mathbf{H} = \mathrm{Ker}\, d / \mathrm{Im}\, d$, *then* $\dim \mathbf{H}$ *is a Legendrian isotopy invariant of* Γ.

1.7. Gradings

It is not known whether Theorem 1.4 alone actually provides new invariants of Legendrian isotopy classes. However, Chekanov's and Eliashberg's invariant has a graded version, which is known to be new. In order to introduce a grading, we need to fix an orientation of Γ, although the degrees $\deg \mathbf{x}$, defined below, do not depend on this orientation. For a point $x \in \Gamma$, let $\Phi(x)$ denote the angle (measured counterclockwise) from the positive direction of the x-axis to the xy-projection of the positive tangent to Γ at x. The function Φ is multivalued; $\Phi(x)$ is defined up to the addition of integral multiples of 2π. If $\mu = 0$, then Φ has a continuous branch $\varphi : \Gamma \to \mathbb{R}$. If $\mu \neq 0$, then Φ has a continuous branch $\varphi : \Gamma \to \mathbb{R}/2\pi|\mu|\mathbb{Z}$. Let $\mathbf{x}$ be a crossing of the xy-diagram of Γ and $x_1, x_2 \in \Gamma$ be the two points above $\mathbf{x}$. Let $\alpha(\mathbf{x})$ be the measure of the positive angle at $\mathbf{x}$ (thus, $0 < \alpha(\mathbf{x}) < \pi$). We also suppose that x_1 lies below x_2, i.e., the z-coordinate of x_1 is less than that of x_2.

Proposition 1.5 (Chekanov [3] and Eliashberg [4]). *The number* $\varphi(x_1) - \varphi(x_2) - \alpha(\mathbf{x})$ *is an integral multiple of* π.

Definition. $\deg(\mathbf{x}) = (1/\pi)(\varphi(x_1) - \varphi(x_2) - \alpha(\mathbf{x}))$.

Obviously,

$$\deg(\mathbf{x}) \in \begin{cases} \mathbb{Z}, & \text{if } \mu = 0, \\ \mathbb{Z}_{2|\mu|}, & \text{if } \mu \neq 0. \end{cases}$$

Using this definition, we introduce a $\mathbb{Z}$- or a $\mathbb{Z}_{2|\mu|}$-grading in $\mathbf{A}$: $\deg(\mathbf{x}_1 \cdots \mathbf{x}_n) = \deg(\mathbf{x}_1) + \cdots + \deg(\mathbf{x}_n)$.

Proposition 1.6 (Chekanov [3] and Eliashberg [4]). *The differential $d : \mathbf{A} \to \mathbf{A}$ is homogeneous of degree -1.*

Proposition 1.6 shows that the cohomology $\mathbf{H}$ is graded: $\mathbf{H} = \oplus_{r\in\mathbb{Z}}\mathbf{H}_r$ or $\mathbf{H} = \oplus_{r\in\mathbf{Z}_{2|\mu|}}\mathbf{H}_r$.

Theorem 1.7 (Chekanov [3] and Eliashberg [4]). *The dimensions* $\dim \mathbf{H}_r$ *are Legendrian isotopy invariants of Γ.*

1.8. An alternative approach

A recent preprint by Ng [14] contains a new construction of (a $\mathbb{Z}$-version of) the Chekanov–Eliashberg algebra in terms of the front diagram. The author considers a free associative $\mathbb{Z}$-algebra generated by the right cusps and all of the crossings of a generic *xz*-diagram of a Legendrian knot. Then, he introduces a differential and proves theorems similar to those above. In particular, Ng's paper provides a grading of the algebra (modulo twice the rotation number): the degree of a cusp is defined as $+1$; the degrees of crossings are also important for this paper and will be described below.

1.9. Augmentations

We will need a coarser invariant, constructed by Chekanov.

Definition. An algebra homomorphism $\varepsilon : \mathbf{A} \to \mathbb{Z}_2$ is called *an augmentation*, if $\varepsilon\circ d = 0$. An augmentation ε is called *graded*, if $\varepsilon(a) = 0$ for any homogeneous $a \in \mathbf{A}$, such that $\deg a \neq 0$. An augmentation is called ρ-graded for some divisor ρ of $2|\mu|$ if $\varepsilon(a) = 0$ whenever $\deg a \not\equiv 0 \bmod \rho$.

Let $\varepsilon : \mathbf{A} \to \mathbb{Z}_2$ be an augmentation. Let $\mathbf{A}_0 = \operatorname{Ker}\varepsilon$. Since $d(a) \in \operatorname{Ker}\varepsilon = \mathbf{A}_0$ for $a \in \mathbf{A}$, then $d(a_1\cdots a_k) = \sum_{i=1}^{k} a_1\cdots a_{i-1}d(a_i)a_{i+1}\cdots a_k \in \mathbf{A}_0^k$ for any $a_1,\dots,a_k \in \mathbf{A}_0$. Thus, $d(\mathbf{A}_0^k) \subset \mathbf{A}_0^k$. In particular, this gives rise to a map $d_\varepsilon : \mathbf{A}_0/\mathbf{A}_0^2 \to \mathbf{A}_0/\mathbf{A}_0^2$ for which $d_\varepsilon^2 = 0$. This is a "linearization" of the differential d.

For a crossing $\mathbf{x}$, we set $\mathbf{x}^\varepsilon = \mathbf{x} + \varepsilon(\mathbf{x}) \in \mathbf{A}$. Obviously, $\mathbf{x}^\varepsilon \in \operatorname{Ker}\varepsilon = \mathbf{A}_0$. Moreover, the correspondence $\mathbf{x} \mapsto \mathbf{x}^\varepsilon$ defines an isomorphism between the vector space $\mathcal{A}$ spanned by the crossings of the *xy*-diagram and $\mathbf{A}_0/\mathbf{A}_0^2$. Hence, d_ε can be regarded as a differential in $\mathcal{A}$. If the augmentation ε is graded, then d_ε is homogeneous of degree -1, and the homology $\mathcal{H}_\varepsilon = \operatorname{Ker} d_\varepsilon/\operatorname{Im} d_\varepsilon$ is $\mathbb{Z}$- or $\mathbb{Z}_{2|\mu|}$-graded; in the ρ-graded case, $\mathcal{H}_\varepsilon$ is $\mathbb{Z}_\rho$-graded.

In practice, $d_\varepsilon : \mathcal{A} \to \mathcal{A}$ is calculated as follows. We take the formula for $d(\mathbf{a})$ and replace all the crossings $\mathbf{x}$ in the polynomial on the right-hand side of the formula by $\mathbf{x}^\varepsilon$. The resulting polynomial will have zero constant term, since $\varepsilon\circ d = 0$. Then, we erase all monomials of degree > 1; the result will be a linear combination of crossings, which is $d_\varepsilon(\mathbf{a})$.

Let I be the set of non-negative integers $\dim \mathcal{H}_\varepsilon$ computed for all possible augmentations $\varepsilon : \mathbf{A} \to \mathbb{Z}_2$, and let I_{gr} be the set of all Poincaré polynomials of $\mathcal{H}_\varepsilon$ computed for all possible graded or ρ-graded augmentations $\varepsilon : \mathbf{A} \to \mathbb{Z}_2$.

Theorem 1.8 (Chekanov [3]). *I and $I_{\rm gr}$ are Legendrian isotopy invariants.*

1.10. Results

The existence of an augmentation or of a (ρ-) graded augmentation, is by itself a Legendrian isotopy invariant of a Legendrian knot. It is not known whether this invariant is determined by the topology of the knot plus its classical invariants. (Actually, this seems likely to me, see the discussion in Section 2.8.) Still, it is important to know whether a (graded) augmentation exists for a given knot.

Theorems 2.1 and 2.2 provide an explicit and easy-to-check sufficient condition for the existence of a graded augmentation in terms of the *xz*-diagram. Using these theorems, we will be able to construct a large quantity of Legendrian knots, for which an augmentation (or a graded augmentation) exists.

Let Γ and Γ' be topologically isotopic oriented Legendrian knots with equal classical invariants. If the existence of a (ρ-) graded augmentation for Γ and Γ' is established, then we have a chance to distinguish them without much work. Suppose, e.g., that we know the numbers of crossings of each degree for some *xy*-diagrams of Γ and Γ'. If, say, the diagram of Γ has at least one crossing of degree k and no crossings of degrees $k \pm 1$, and the diagram of Γ' has no crossings of degree k, then the corresponding homologies $\mathcal{H}$ must be different in dimension k, and the knots Γ and Γ' cannot be Legendrian isotopic. (This simple observation alone is sufficient to distinguish the knots in the Chekanov–Eliashberg example.) Some similar possibilities are discussed in Section 2.7.

Proof of Theorems 2.1 and 2.2 is contained in Section 3. These proofs are based on a construction which may be useful for other problems arising in Chekanov–Eliashberg theory. For a very brief description of the main idea, I can say that for a computation of Chekanov–Eliashberg invariant it seems preferable to deal with an *xy*-diagram with as few crossing as possible. We do the opposite: we transform a diagram in such a way that it gains enormously many new crossings, the more, the better. However, in this way we achieve some taming of the differential.

2. Existence of augmentations

2.1. Rulings

Consider an *xz*-diagram of a Legendrian knot. Obviously, the numbers of left and right cusps are the same. A *ruling* of the diagram consists, by definition, of

(a) a 1-1 correspondence between left and right cusps;
(b) two paths within the diagram joining each pair of corresponding cusps.

This correspondence and these paths should satisfy the following conditions:

(1) the paths joining a pair of corresponding cusps are disjoint (which makes one of them *upper* and another one *lower*);

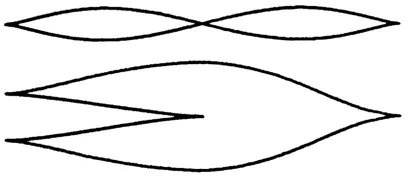

Fig. 1.

(2) any two paths of the ruling can meet only at the crossings of the diagram.

Conditions (1) and (2) imply:

(3) the paths of the ruling cover the diagram;

(4) no path ever passes through a cusp (in other words, the x-coordinate is monotonic on each path of the ruling).

Not every xz-diagram of a Legendrian knot can be ruled; examples of diagrams without rulings are given in Fig. 1. (Any diagram can be made unrullable by adding a Z-shaped zigzag or a crossing next to a cusp, formed from the two strands of the cusp.) A diagram can have more than one ruling, e.g., the diagram in Fig. 2 has at least four different rulings.

In order to visualize a ruling one can shadow the domains between the paths joining the same pair of cusps; it should be taken into account, however, that these domains may overlap.

2.2. *Normal rulings*

From now on, we consider only xz-diagrams which satisfy the following additional *genericity* condition: no two crossings or cusps have the same x-coordinate.

Consider a ruled (generic) xz-diagram. Let a be a crossing. Then precisely two paths of the ruling pass through a; let them be p_1 and p_2. We say that a is a *switch*, if p_1 and p_2 exchange strands at a; in other words, a is a switch, if, in some neighborhood of a, excluding a, one of the paths p_1, p_2, say p_1, goes above the other one. Since p_1 and p_2 share a crossing, they cannot join the same pair of cusps (see Condition (1)). Let q_1 and q_2 be the paths of the ruling that join the same pairs of cusps as, respectively, p_1 and p_2. Since

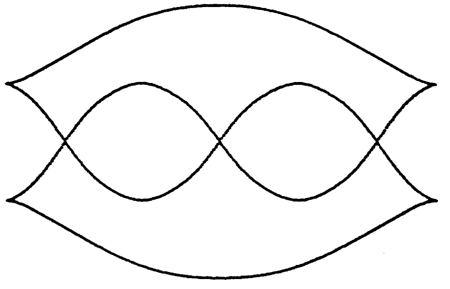

Fig. 2.

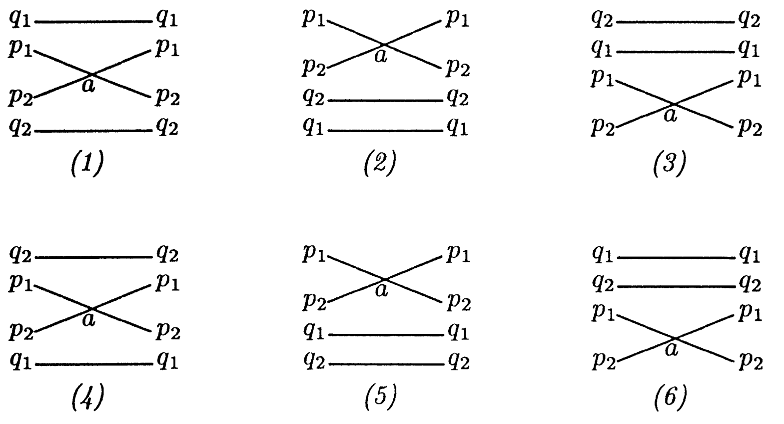

Fig. 3.

our diagram is generic, there are six possible arrangements of the paths p_1, p_2, q_1, q_2 in a neighborhood of the vertical line passing through a, see Fig. 3.

In cases (1)–(3), we call the switch *normal*; in cases (4)–(6), we call the switch *abnormal*. A ruling is called normal, if all its switches are normal.

2.3. Existence of augmentations

Theorem 2.1. *If the xz-diagram of a Legendrian knot admits a normal ruling, then this Legendrian knot has an augmentation.*

See the proof in Section 3.

2.4. Rulings and gradings

For a crossing a of an xz-diagram, consider a path along the diagram (without switching strands at crossings) from a back to a, starting along the strand with the bigger slope (in any direction); there are two such paths. Let d and u be the numbers of downward and upward cusps on our paths. The residue $d - u \bmod 2|\mu|$, where μ is the Maslov number of the knot, does not depend on the choice of the path; we call this residue the *degree* of a. (This definition coincides with that in [14], see 1.8.)

A ruling is called graded, if all the switch crossings have degree 0. It is called ρ-graded, if the degrees of the switch crossings are divisible by ρ. These properties of a ruling do not depend on the choice of orientation.

2.5. Existence of graded augmentations

Theorem 2.2. *If the xz-diagram of a Legendrian knot admits a graded (or ρ-graded) normal ruling, then this Legendrian knot has a graded (or ρ-graded) augmentation.*

See the proof in Section 3

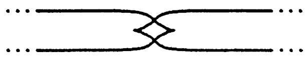

Fig. 4.

2.6. Example 1: doubling of a Legendrian knot

The following construction was shown to me by Yasha Eliashberg in 1997, in connection with the Chekanov–Eliashberg invariant.

2.6.1. Construction. Thurston–Bennequin and Maslov numbers

Take the xz-diagram of an arbitrary Legendrian knot Γ and shift it slightly down. Then replace a small fragment of a pair of parallel strands by a "lock", as shown in Fig. 4. The resulting Legendrian knot will be denoted as Γ_{db1}. A slightly more general construction uses, instead of a lock, a more complicated insert with $2k+2l+2$ cusps, as shown in Fig. 5; here k and l are arbitrary positive numbers. (This insert contains a lock.)

The resulting Legendrian knot is denoted as $\Gamma_{\mathrm{dbl}}(k,l)$. Obviously, $\Gamma_{\mathrm{dbl}}(0,0) = \Gamma_{\mathrm{dbl}}$, and if k and l are both even, then $\Gamma_{\mathrm{dbl}}(k,l)$ can be described as Γ'_{dbl} for some Γ'. Also note that $\Gamma_{\mathrm{dbl}}(k,l)$ and $\Gamma_{\mathrm{dbl}}(k',l')$ are topologically isotopic if and only if $k+1 = k'+l'$.

These constructions are illustrated by Fig. 6, where the standard Legendrian trefoil (shown in Fig. 2) plays the role of Γ.

Proposition 2.3.

$$\mu(\Gamma_{\mathrm{dbl}}(kl)) = 0 \quad \textit{always}, \qquad \tau\beta(\Gamma_{\mathrm{dbl}}(k,l)) = \begin{cases} +1, & \textit{if } k+l \textit{ is even}, \\ -3, & \textit{if } k+l \textit{ is odd}. \end{cases}$$

In particular, for any Γ,

$$\mu(\Gamma_{\mathrm{dbl}}) = 0, \qquad \tau\beta(\Gamma_{\mathrm{dbl}}) = +1.$$

Proof. Direct computation. □

2.6.2. Degrees of crossings in the xz-diagram

In the xz-diagram of $\Gamma_{\mathrm{dbl}}(k,l)$, we will distinguish three types of crossings:

(1) two crossings within the lock;
(2) four crossings originating from each crossing of Γ;

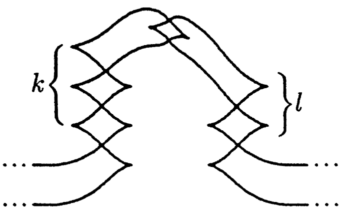

Fig. 5.

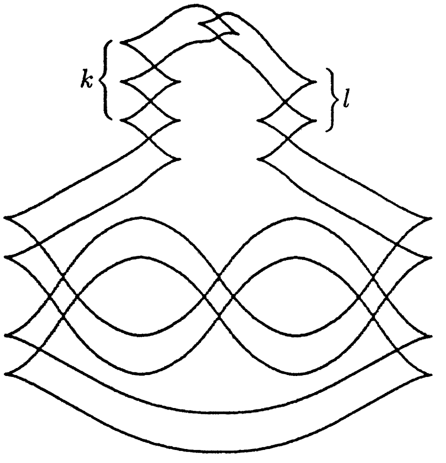

Fig. 6.

(3) the remaining crossings, one near each cusp of Γ, and $2k+2l$ near the lock (but outside the lock).

Proposition 2.4. *All of the crossings of type* (3) *have degree* 0. *The degrees of crossings of type* (2) *do not depend on k or l. Each crossing of type* (1) *has degree equal to* $\pm(\mu(\Gamma)+k-l)$. (*We orient* Γ *in such a way that the k cusps marked in Fig.* 5 *precede the site of the lock, while the l cusps succeed this site*.)

Proof. Again, this is obvious. Note that the degrees of the four crossings near a crossing a of Γ are $r-1, r, r, r+1$, where r is the integral degree of a calculated according to the procedure described in Section 2.4, using the path avoiding the site of the lock. In particular, the absolute values of these degrees can never exceed the total number of crossings of Γ, or even the maximum of the numbers of upward and downward cusps plus one. □

2.6.3. Degrees of crossings in the xy-diagram

The *xy*-diagram of the knot $\Gamma_{\mathrm{dbl}}(k,l)$ consists of two identical copies of the *xy*-diagram of Γ (we should slightly perturb one of these copies to make the diagram generic) with the insert shown in Fig. 7 (compare with [3] or [8]). The following proposition will be referred to in Section 2.7 (we will be able to avoid it if we use the result of [14]).

Proposition 2.5. *The degrees of the crossings a of Fig.* 7 *are* $\pm(\mu(\Gamma)+k-l)$, *the degrees of the crossings b are all equal to* 1, *the degrees of the crossings c are all equal to* 0, *the degrees of the crossings outside Fig.* 7 *do not depend on k or l.*

Proof. This is obvious from the diagram. Note that outside Fig. 7, there will be two kinds of crossings: four crossings near each crossing of the *xy*-diagram of Γ and some amount of

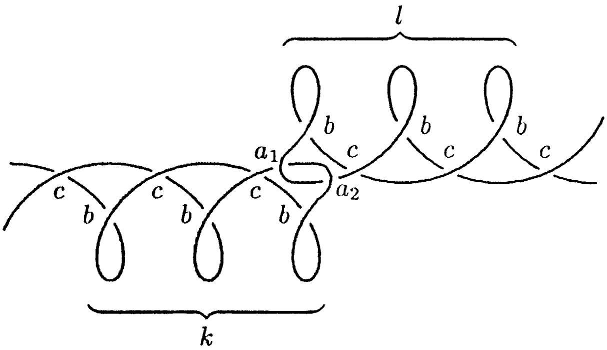

Fig. 7.

accidental crossings between the two copies of this diagram. The degrees of the accidental crossings are 0 or 1 (there is a path from such a crossing back to itself that goes to the lock and then back along almost the same track; whether the degree is 0 or 1, depends on the location of the positive angle). The degrees of crossings near a crossing of Γ are the same as that of the crossing of Γ, maybe plus or minus one. □

2.6.4. Existence of augmentation

Theorem 2.6. *For any Γ, k, l, the Legendrian knot $\Gamma_{\mathrm{dbl}}(k,l)$ possesses a graded augmentation.*

Proof. According to Theorem 2.2, it is sufficient to find a graded normal ruling for the xz-diagram of $\Gamma_{\mathrm{dbl}}(k,l)$. Such a ruling is presented in Fig. 8. Remark: the two cusps of the

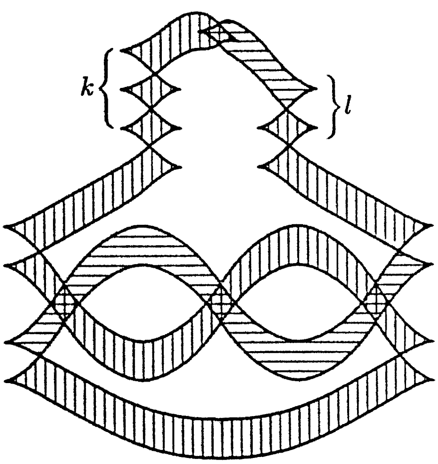

Fig. 8.

lock correspond not to each other, but rather to the two nearest cusps to the left and to the right of the lock. We choose this correspondence to avoid switches at the crossings within the lock (because they have non-zero degrees). In our construction, all the switches occur at a crossing of type (3), all of which have zero degree (Proposition 2.4). Also note that all the switches of this ruling belong to type (1) of Fig. 3. □

2.7. *Distinguishing Legendrian knots*

Let Γ_1 and Γ_2 be two Legendrian knots. Suppose that (1) they are topologically isotopic, (2) they have equal Thurston–Bennequin and Maslov numbers, (3) they have graded (or ρ-graded) augmentations. According to Theorem 1.8, if Γ_1 and Γ_2 are Legendrian isotopic, then for any graded augmentation of Γ_1, there exists a graded augmentation of Γ_2 such that the (graded) homologies of the linearized Chekanov–Eliashberg complexes are the same. Sometimes, we can use this to distinguish Γ_1 and Γ_2 without knowing the differential in these complexes. Indeed, let c_r be the number of crossings of degree r in the xy-diagram of Γ_1, and let d_r be the similar number for Γ_2. Then the corresponding linearized complexes are

$$C = \{\cdots \to C_{-1} \to C_0 \to C_1 \to \cdots\}, \quad D = \{\cdots \to D_{-1} \to D_0 \to D_1 \to \cdots\}$$

with $\dim C_r = c_r$, $\dim D_r = d_r$, (where dim is the dimension over $\mathbb{Z}_2$). As it was noted in 1.10, the equality of homologies, $H(C) = H(D)$, is impossible, say, if, for some k, $c_k \neq 0$, $c_{k-1} = c_{k+1} = d_k = 0$ (which is, actually, sufficient for Theorem 2.8). More generally, the equality $H(C) = H(D)$ implies, at least in the $\mathbb{Z}$-graded case, a chain of Morse-like inequalities, each of which becomes a tool for distinguishing Legendrian knots.

Proposition 2.7. *Let C and D be $\mathbb{Z}$-graded, and let $c_r = d_r = 0$ for $r < m$. For $k \geq m$, let $e_k = c_m - c_{m+1} - c_{m+2} + \cdots \pm c_k$, $f_k = d_m - d_{m+1} - d_{m+2} + \cdots \pm d_k$. Then*

$$e_m \geq f_{m+1}, \quad e_{m+1} \leq f_{m+2}, \quad e_{m+2} \geq f_{m+3}, \ldots$$
$$f_m \geq e_{m+1}, \quad f_{m+1} \leq e_{m+2}, \quad f_{m+2} \geq e_{m+3}, \cdots .$$

The proof has been well known for at least 75 years.

Theorem 2.8. *For any Legendrian knot Γ, there exists an integer N with the following property. If for some integers k, l, k', l', the Legendrian knots $\Gamma_{\mathrm{dbl}}(k, l)$, $\Gamma_{\mathrm{dbl}}(k'l')$ are Legendrian isotopic, and $|k - l| > N$, then either $k = k'$, $l = l'$, or $k = l' - \mu(\Gamma)$, $l = k' + \mu(\Gamma)$.*

The proof is based not on Proposition 2.7, but rather on the trivial remarks before it. Since our knots are Legendrian isotopic, they are also topologically isotopic, and hence $k + l = k' + l'$ (see the remark in Section 2.6.1). According to Theorem 2.6, both knots have graded augmentations. The linearized complex for $\Gamma_{\mathrm{dbl}}(k, l)$ has non-zero components of degrees $\pm(\mu(\Gamma) + k - l)$, and, if N is big enough, at least one of these components is isolated (the two neighboring components are trivial). Hence, the complex for $\Gamma_{\mathrm{dbl}}(k', l')$ should also have a component of this degree. This is possible only if $\mu(\Gamma)+k-l=\pm(\mu(\Gamma)+l'-l')$, which, in combination with $k+l=k'+l'$, implies the statement of the theorem.

Remarks.

1. Since for any m, all Legendrian knots $\Gamma_{\rm dbl}(k,l)$ with $k+l=m$ are topologically isotopic and have the same classical invariants, Theorem 2.8 produces a lot of examples of topologically isotopic Legendrian knots, not distinguishable by the classical invariants.
2. The simplest of these knots, corresponding to Γ with the lens-like *xz*-diagram (two cusps and no crossings) and $k+l=4$, is Eliashberg's and Chekanov's initial example. The same example with arbitrary k and l was considered in the later version of [3] (see also [8]).
3. We do not discuss here how big N should be (and whether it is needed at all). From the remarks after Propositions 2.4 and 2.5, it is not hard to deduce that any N greater than both the number of downward cusps and the number of upwards cusps (in the *xz*-diagram) will work. A little additional work allows us to replace this "greater than" by "greater than or equal to".
4. Theorem 2.2 and Proposition 2.7 can probably be used for constructing many more examples.

2.8. Example 2: mirror torus knots. A conjecture

Let $p>q$, where $\mathrm{GCD}(p,q)=1$, and let $M(p,q)$ be the topological isotopy class of the mirror torus knot that winds p times in the direction of a meridian and q times in the direction of a parallel. It is proved in [10] that Legendrian knots of type $M(p,q)$ with equal Thurston–Bennequin and Maslov numbers are Legendrian isotopic; moreover, the authors determine all possible values of these numbers within the topological class $M(p,q)$. Thus, the problem of Legendrian isotopy classification of Legendrian mirror torus knots has been completely solved. Still one may wonder whether a (graded) augmentation exists for these knots, and the answer to this question looks unexpected and gives a push to some amazing speculations.

Most of the results mentioned in this section are contained in a paper [9].

The *xz*-diagram in Fig. 9 (borrowed from Epstein's paper [7]) with p cusps on the left and q cusps on the right (plus $p-q$ cusps in the middle) represents a Legendrian knot $L(p,q)$ of the topological type $M(p,q)$ (in Fig. 9, $p=7, q=4$). A simple calculation shows that

$$\tau\beta(L(p,q))=-pq, \qquad \mu(L(p,q))=\pm(p-q)$$

(depending on the orientation). A maximal possible value of $-pq$ for the Bennequin invariant of type $M(p,q)$ was first established for odd q in [7] and subsequently for any q in [10].

Proposition 2.9. *If q is even, then the Legendrian knot $L(p,q)$ has a $(p-q)$-graded augmentation.*

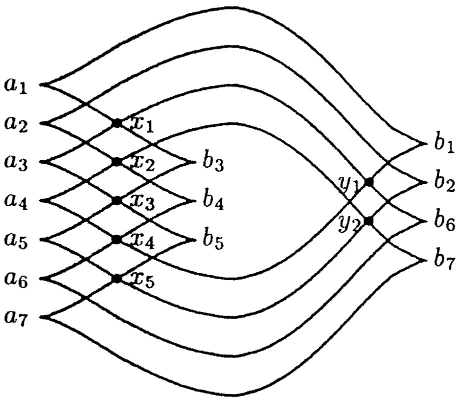

Fig. 9.

Proof. Let $q = 2r$. We label the cusps of the left vertical row as $a_1, \dots, a_p$ (from the top to the bottom), the cusps of the middle vertical row as $b_{r+1}, \dots, b_{p-r}$, and the cusps of the right vertical row as $b_1, \dots, b_r, b_{p-r+1}, \dots, b_p$ (as shown in Fig. 9). The crossings of the diagram are arranged in two groups of vertical rows, $2r - 1$ rows in each; the rows of the left group contain, respectively, $p - 1, \dots, p - 2r + 1$ crossings; the rows of the right group contain, respectively, $1, \dots, 2r - 1$ crossings. We label the crossings of the middle row in the left group as $x_1, \dots, x_{p-r}$ and the crossings of the middle row in the right group as $y_1, \dots, y_r$ (from the top to the bottom).

Consider the ruling of the diagram that assigns the cusp b_i to the cusp $a_i (i = 1, \dots, p)$ and consists of the paths

$$\begin{array}{ll}
a_i \left\{ \begin{array}{l} \text{—————————————} \\ \text{——— } x_i \text{ ————— } y_i \text{ ———} \end{array} \right\} b_i & i = 1, \dots, r, \\[1ex]
a_i \left\{ \begin{array}{l} \text{—————— } x_{i-r} \text{ ——————} \\ \text{—————— } x_i \text{ ——————} \end{array} \right\} b_i & i = r+1, \dots, p-r, \\[1ex]
a_i \left\{ \begin{array}{l} \text{—— } x_{i-r} \text{ —— } y_{i-(p-r)} \text{ ——} \\ \text{—————————————} \end{array} \right\} b_i & i = p-r+1, \dots, p;
\end{array}$$

half of them is shown in Fig. 10.

To complete the proof, we notice that all the crossings x_i, y_i have the "middle" degree $p - q$. $\square$

The fact that q is even is essential in this proof. Actually, for q odd, the result is the opposite.

Proposition 2.10 (Epstein and Fuchs [9]). *If q is odd, then the Legendrian knot $L(p, q)$ admits no augmentations (graded or not).*

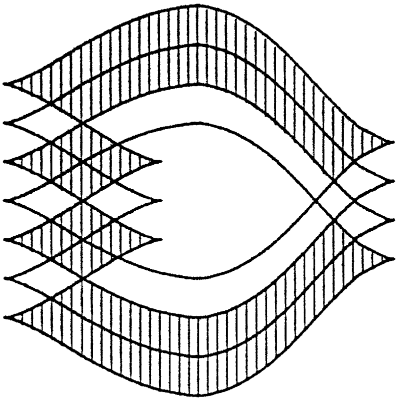

Fig. 10.

This strange difference between the cases of odd and even q arises again in a computation of entirely different nature. Recall that, according to Theorem 1.3, for any Legendrian knot L,

$$\tau\beta(\Gamma) < -f(\Gamma), \tag{1}$$

where f denotes the highest degree of the Kauffman polynomial in the variable z.

A computation (based on the results of [18]) gives the following result.

Proposition 2.11 (Epstein [7] and Epstein and Fuchs [9]).

$$f(M(p,q)) = \begin{cases} pq, & \textit{if } q \textit{ is even}, \\ pq - p + q, & \textit{if } q \textit{ is odd}. \end{cases}$$

Thus, the estimate of Theorem 1.2 is sharp within the topological type $M(p,q)$ if and only if q is even.

Let us try to guess when a Legendrian knot possesses an augmentation. It looked plausible that the right condition for that is the maximality of the Thurston–Bennequin number within the topological isotopy class; however, Proposition 2.10 shows that this condition is at least not sufficient. Propositions 2.9–2.11 together give rise to the following.

Irresponsible conjecture. *A (graded?) augmentation for a Legendrian knot L exists if and only if* $\tau\beta(L) = -f(L) - 1$.

3. Proofs

3.1. Introduction

As it was explained in Section 1.9, an augmentation for the Chekanov–Eliashberg algebra of a generic *xy*-diagram of a Legendrian knot may be regarded as a function ε on the set of crossings of the diagram with values in $\mathbb{Z}_2$ and with an additional property described below. Let $\mathbf{x}$ be a crossing, and let $f \in \mathrm{Imm}_n(\mathbf{x})$ (see Section 1.6 for the notation). We call an immersion f special (with respect to ε), if $\varepsilon(f(u_i)) = 1$ for $i = 1, \dots, n$ (again, we use the notation of Section 1.6). The following definition is equivalent to that given in Section 1.9.

Definition. A function ε is an augmentation, if for every $\mathbf{x}$, the number of special immersions in $\mathrm{Imm}(\mathbf{x}) = \cup_{n \geq 0} \mathrm{Imm}_n(\mathbf{x})$ is even.

For a fixed ε (satisfying the additional property above or not) we call crossings $\mathbf{x}$ with $\varepsilon(\mathbf{x}) = 1$ *marked*, and, graphically, we will encircle marked crossings in *xy*-diagrams.

An augmentation is graded (ρ-graded), if all the marked crossings have degree 0 (degree divisible by ρ).

For example, let us try to take $\varepsilon \equiv 0$ (no marked crossings). Then $f \in \mathrm{Imm}_n(\mathbf{x})$ is special if and only if $n = 0$, and the condition above means that the cardinality $\#(\mathrm{Imm}_0(\mathbf{x}))$ of the set $\mathrm{Imm}_0(\mathbf{x})$ is even for every $\mathbf{x}$. This is not impossible (it is the case, e.g., for a figure-eight-shaped diagram with one crossing; I doubt that there are other examples), but is not common. If $\#(\mathrm{Imm}_0(\mathbf{x}))$ is odd for some $\mathbf{x}$, then we have to mark some (other) crossings. This suggests a strategy for constructing augmentations: we should first look at immersions of P_0. Another important thing is that the existence of (graded, ρ-graded) augmentations is a Legendrian isotopy invariant, so we have the right to modify our diagram within a Legendrian isotopy class as we wish.

3.2. A modification of a diagram

Consider a generic *xy*-diagram of a Legendrian knot. We suppose that the values of the x-coordinates of crossings of this diagram, as well as the cusps and crossings of the *xz*-diagram of the same knot are all different. We call these values of the x-coordinate singular. (The cusps of the *xz*-diagram correspond to the sites of vertical tangents of the *xy*-diagram.)

Suppose that the whole *xy*-diagram is located within a horizontal strip $L < y < M$. We apply to our diagram a number of similar elementary modifications, which we describe below.

We take a tiny segment $[a, b]$ of the diagram, no point of which has a singular x-coordinate. Suppose that a is located to the left of b. Then replace this segment by the curve consisting of five straight segments: from a vertically up, to the level $y = M$; then a very short way horizontally to the right; then vertically down, between a and b, to the level $y = L$; then again a short way horizontally to the right; then vertically up to the point b. (We can round the corners to make the curve smooth, but it is not really necessary.) The modification is shown in Fig. 11. We will refer to the 5-gonal insert of Fig. 11 as a *splash*. Certainly, we can choose the sizes of its horizontal parts in such a way that the total gain of the area will be zero. Note that the z-coordinate remains almost constant on a splash; in the *xz*-diagram, splashes correspond to very low and even more narrow and sharp thorns directed up.

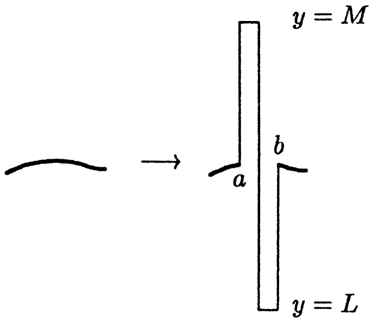

Fig. 11.

Now, we add to our *xy*-diagram a very large number of disjoint splashes, in such a way that on each strand they follow each other with a very short step (although considerably exceeding the width of a splash), and between any two singular values of x the number of splashes on each strand considerably exceeds the number of strands. It is convenient (although not necessary) to arrange splashes in the following order. Slightly to the right of a singular value of x, we make a splash at the top strand; then we move slightly to the right and make a splash on the second top strand, and so on; when we reach the bottom strand, we jump back to the top strand and begin a new descent.

The result (for a very simple *xy*-diagram) is shown in Fig. 12. Note that in reality the number of splashes should be much bigger, and that Fig. 12 ignores the crossings of the *xz*-diagram. Also note that the shadowed domains of Fig. 12 will be relevant in Section 3.4. Below, we will consider the modified diagram (with splashes), but when we speak of singular values of x, we will mean the initial, unmodified diagram.

3.3. The sets Imm

The abundance of crossings of the modified diagram is compensated, at least partially, by rather a simple structure of immersions considered in Section 1.6. Let $\mathbf{x}$ be a crossing of the diagram, and let $f : P_n \to \mathbb{R}^2$ be an immersion from $\mathrm{Imm}_n(\mathbf{x})$. Then the (counterclockwise oriented) closed curve $f(\Pi_n)$ goes along the diagram with $n + 1$ left turns at crossings, and its (inner) left hand side is never exposed to the outer domain of the diagram.

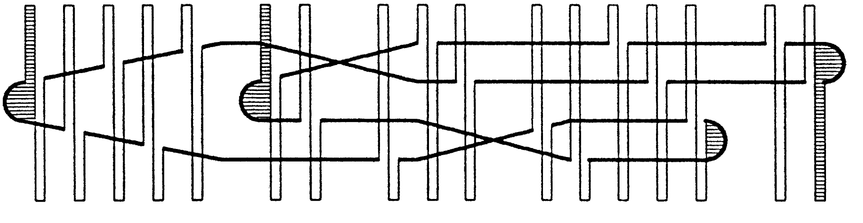

Fig. 12.

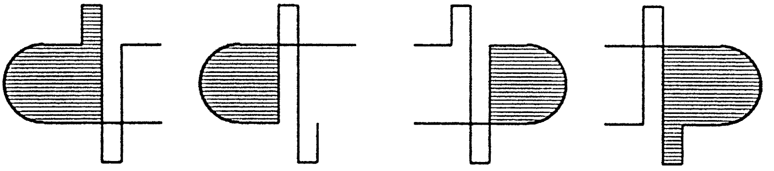

Fig. 13.

This shows that it can never contain a whole splash, since both sides of a splash are exposed to the outer domain (see Fig. 11). Also, it can never travel very far from **x** in the horizontal direction: at any turn, besides the turn at **x**, the value of the z-coordinate decreases, and the turns should be at least as close to each other as splashes on the same strand; but at any strand (including the splashes) the value of the z-coordinate is almost constant, so if the value of the x-coordinate of **x** is sufficiently far from the singular values, then the total number of turns cannot exceed the number of strands; the case when the x-coordinate of **x** is dangerously close to a singular value, is only slightly different. Actually, the special ordering of splashes, described in Section 3.2, makes the allowed immersions still simpler.

Proposition 3.1.

(1) *If* $n > 3$, *then* $\mathrm{Imm}_n(\mathbf{x})$ *is empty for any* **x**.
(2) *If* $f \in \mathrm{Imm}_n(\mathbf{x})$, *then f is an embedding*.

Technically, we will not need Proposition 3.1, and we leave its proof to the reader.

3.4. The sets Imm$_0$

Let **c** be a point of the initial diagram (without splashes) with a vertical tangent. Then, to the left or to the right of **c**, there are two strands issuing from **c**; we denote them as s and s'. Of the splashes on s and s', take the one closer to **c**. Let **x** be the intersection point of this splash and the other of the strands s and s'; then $\mathrm{Imm}_0(\mathbf{x})$ contains a well-visible element: an embedded disk bounded by a segment of the splash and the segments of the strands s and s'. These disks are shadowed in Fig. 12. There are four slightly different possible appearances of such a disc; they are shown in Fig. 13.

Proposition 3.2. *The construction above gives all elements of* $\mathrm{Imm}_0(\mathbf{x})$ *for all* **x**; *in particular, there is a* 1-1 *correspondence between the set* $\mathrm{Imm}_0 = \cup_{\mathbf{x}}\mathrm{Imm}_0(\mathbf{x})$ *and the set of cusps of the xz-diagram*.

(There may be some splashes on the strands not shown in Fig. 13 crossing the shadowed domains.)

Proof. For $f \in \mathrm{Imm}_0(\mathbf{x})$, the boundary curve $f(\Pi_0)$ of $f(P_0)$ contains, but is not restricted to, a part of a splash. Hence, the crossing **x** has to be a crossing of a strand and a crossing

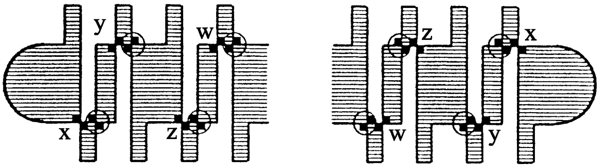

Fig. 14.

on the same strand, which forces the strand to make a U-turn next to the splash. All the possibilities for that are shown in Fig. 13. □

3.5. Construction of an augmentation from a normal ruling

On all further diagrams, we mark the positive angles by small black squares.

3.5.1. Cusps

Of the four diagrams of Fig. 13, we consider only two; the remaining two are absolutely similar. When we refer to Fig. 14, we mean any one of the diagrams presented. Note that on the left of these diagrams, the strand that is the upper one in the *xy*-diagram is also the upper one in the *xz*-diagram; on the contrary, in the right diagram the strand that is the upper one in the *xy*-diagram becomes the lower one in the *xz*-diagram.

Near a cusp, we have a crossing $\mathbf{x}$ with $\#(\mathrm{Imm}_0(\mathbf{x})) = 1$. To compensate for this, we observe that $\mathrm{Imm}_1(\mathbf{x})$ contains an immersion (embedding) of P_1, whose image is shadowed in Fig. 14: the bottom part or the top part of the splash nearest to the cusp. So, if we mark the image of u_1 with respect to this immersion (as done in Fig. 14), then the number of special immersions in $\mathrm{Imm}(\mathbf{x})$ becomes even (two). But the marked crossing also belongs to the boundary of the next shadowed disk, which makes the number of special immersions odd (one) in $\mathrm{Imm}(\mathbf{y})$. This forces us to mark one more crossing (next to $\mathbf{y}$), and so on.

Assume now that the *xz*-diagram of the Legendrian knot considered is normally ruled. Then we mark the crossings along the paths emanating from each cusp, as described above, until we reach the next singular value of x.

Note that even if there are several pairs of such paths, one over another, only immersions shown in Fig. 14 will be special, which is seen from the following. Each marked crossing involves only strands belonging to one pair, and all the pairs are disjoint. If $f \in \mathrm{Imm}_n(\mathbf{x})$, then $f(u_i), i = 1, \dots, n$ are crossings between strands s_i, s_{i+1} for a certain sequence of strands $s_1, \dots, s_{n+1}$. Thus, if $n \geq 2$, then the pairs cannot be disjoint, and if $n = 1$, then the immersion f involves two strands from the same pair, and all such immersions are shown in Fig. 14.

3.5.2. Crossings

Crossings between the strands from different pairs, either in the *xy*-diagram or in the *xz*-diagram, are irrelevant: they do not interfere with the picture in Fig. 14.

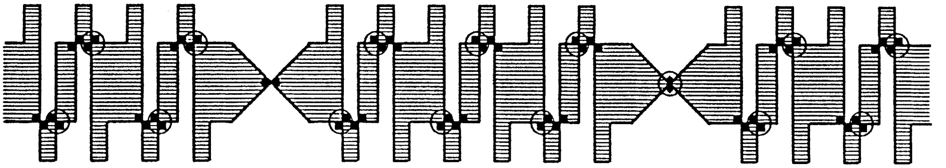

Fig. 15.

If a crossing between two paths of the ruling emanating from the same cusp occurs, then the construction in Fig. 14 may be continued as shown in the diagram in Fig. 15. In this diagram, all the special immersions are shadowed; all the marked crossings are encircled, and for any crossing **x** that is not encircled, the number of special immersions in Imm(**x**) is two.

Although we do need this for the proof, the reader may consider the *xy*-picture corresponding to a crossing of the two paths as above in the *xz*-diagram (Fig. 16). This occurrence (forbidden by the definition of a ruling) presents an unsurmountable obstacle for our construction.

The concluding remark of Section 3.5.1 remains valid.

3.5.3. Switches

Four strands, falling into two pairs, are involved in a switch. Two of them, belonging to the different pairs, form a crossing in the *xz*-diagram. The position of the other two strands in the *xy*-diagram many be chosen arbitrarily (we can make them wavy in the *xz*-diagram, the resulting crossings in the *xy*-diagram do not matter, as was explained above. To avoid considering many cases, we can assume that these two strands cross each other in the *xy*-diagram at the same value of the *x*-coordinate. In the three cases of normal switches (see Section 2.2), we can make the relevant part of the *xz*-diagram look like one of the three diagrams in Fig. 17.

In all three cases the pairs to the left of the switch are $a \leftrightarrow b$, $c \leftrightarrow d$, and the pairs to the right of the switch are $a \leftrightarrow c$, $b \leftrightarrow d$. Cases (2) and (3) are absolutely similar. The *xy*-diagrams (modified by splashes) corresponding to cases (1) and (3) are shown in Figs. 18 and 19.

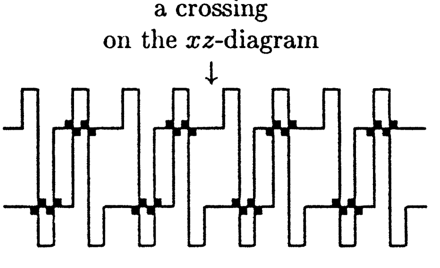

Fig. 16.

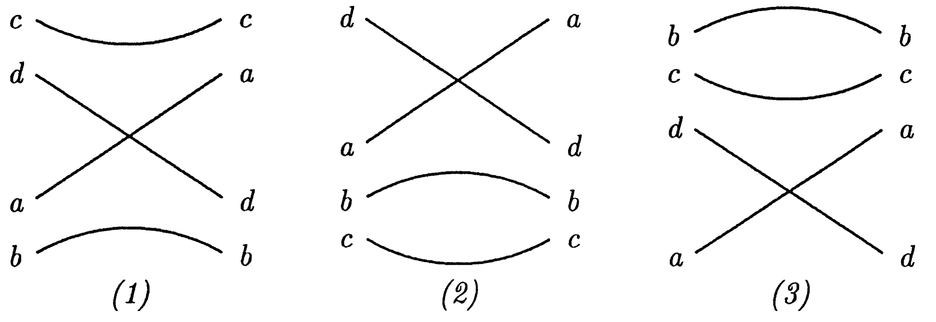

Fig. 17.

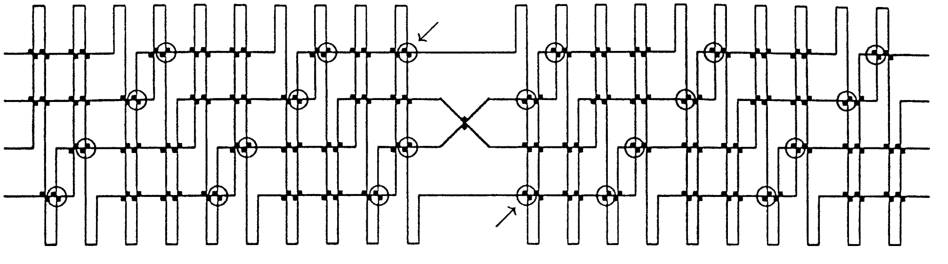

Fig. 18.

Note that in the cases, both to the left and to the right of the switch, the pairs of the strands in the *xy*-diagram corresponding to the pairs of paths of the ruling are the two upper strands and two bottom strands. The augmentation at a distance from a switch is defined in accordance with Figs. 14 and 15. To complete the definition of the augmentation in the proximity of the switch, we need some additional markings, which are shown by arrows on Fig. 18.

It is worth mentioning that, as one may expect, the corresponding diagrams for abnormal switches are resistant to the constructions similar to those shown in Figs. 18 and 19.

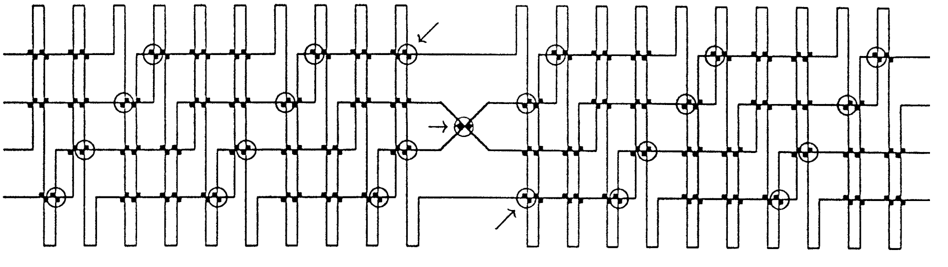

Fig. 19.

3.5.4. Gradings

If the ruling is graded or ρ-graded, then the augmentation constructed above is also graded or ρ-graded. To demonstrate this, we need to calculate the degrees of the marked crossings, which is easy to do directly from the definitions.

4. Note added in proof

A notion of a ruling, without the normality condition, was first considered by Eliashberg in 1987 (Y. Eliashberg, A theorem on the structure of wave fronts, and its applications in symplectic topology, Funct. Anal. and Appl., 1987, vol. 21). Eliashberg proved the existence of a ruling for a Legendrially unknotted Legendrian knot. In 2000, Chekanov and Pushkar, independently of me, introduced, in connection with the Arnold four-cusp conjecture, a notion equivalent to my graded normal ruling (Y. Chekanov and P. Pushkar, The combinatorics of Legendrian knot fronts, Arnold conjecture and Legendrian knot invariants (in Russian), preprint). They proved that the existence of a graded normal ruling, and even the number of different graded normal rulings, are Legendrian isotopy invariants. The relations between rulings and augmentations in the Chekanov–Eliashberg algebra has been never considered by these authors.

Recently, T. Ishkhanov and myself proved that the sufficient condition of Theorems 2.3.1 and 2.5.1 of this paper are also necessary, that is, the existence of a (graded, ρ-graded) normal ruling is equivalent to the existence of a (graded, ρ-graded) augmentation (our work is currently in preparation).

According to P. Pushkar, the existence of a graded normal ruling is also equivalent to the existence of a generating family of functions for a Legendrian knot, which makes the latter equivalent to the existence of an augmentation (Pushkar's work is also in preparation). Pushkar has also a construction, from a generating family of functions, of a Morse-like complex, and it seems to me very likely that this complex is the same as the augmented Chekanov–Eliashberg algebra.

Acknowledgements

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References

[1] D. Bennequin, Entrelacements et équations de Pfaff, Astérisque 107/108 (1983) 87–161.

[2] S. Chmutov, V. Goryunov, Knot polynomials and invariants of Legendrian knots, in: Topics in Singularity Series, AMS Translations, Ser. 2, Vol. 180, 1997, pp. 25–43.

[3] Yu. Chekanov, Differential algebras of Legendrian links, Fields Institute Preprint, 1997; Corrected Version, 1999.

[4] Ya. Eliashberg, Invariants in contact topology, in: Proceedings of the International Congress on Mathematics, Vol. II, Document on Mathematics, 1998, Extra Vol. II, pp. 327–338 (electronic).
[5] Ya. Eliashberg, M. Fraser, Classification of topologically trivial Legendrian knots, Geometry, Topology, Dynamics (Montreal, PQ, 1995), CRM Proceedings, Lecture Notes, Vol. 15, AMS, 1998, pp. 17–51.
[6] Ya. Eliashberg, A. Givental, H. Hofer, Introduction to Symplectic Field Theory, Geom. Funct. Anal., 2000, Special Issue, Part-II, pp. 560–673.
[7] J. Epstein, On the invariants and isoiopies of Legendrian and transverse knots, Dissertation, Davis, 1997.
[8] J. Epstein, D. Fuchs, M. Meyer, Chekanov–Eliashberg invariants and transverse approximations of Legendrian knots, Pac. J. Math. 201 (2001) 89–106.
[9] J. Epstein, D. Fuchs, Invariants of Legendrian mirror torus knots, in: Symplectic Interactions, Fields Inst. Commun. 35 (2003) 105–115.
[10] J. Etnyre, K. Honda, Knots and contact geometry, arXiv:math.GT/0006112.
[11] E. Ferrand, On Legendrian knots and polynomial invariants, arXiv:math.GT/0002250.
[12] D. Fuchs, S. Tabachnikov, Invariants of Legendrian and transverse knots in the standard contact space, Topology 36 (1997) 1025–1053.
[13] L. Kauffman, On knots, Princeton University Press, Princeton, NJ, 1987.
[14] L. Ng, On invariants of Legendrian knots, arXiv:math.GT/0011265.
[15] J. Swiatkowski, On the isotopy of Legendrian knots, Ann. Global Anal. Geom. 10 (1992) 195–207.
[16] S. Tabachnikov, Calculation of the Bennequin invariant of a Legendrian curve from the geometry of its wave front, Funct. Anal. Appl. 22 (1988) 246–248.
[17] S. Tabachnikov, Estimates for the Bennequin number of Legendrian links from state models for knot polynomials, Math. Res. Lett. 4 (1997) 143–156.
[18] Y. Yokota, The Kauffman polynomial of periodic knots, Topology 32 (1993) 309–324.